A trace partitioned Gray code for $q$-ary generalized Fibonacci strings

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Abstract

We provide a trace partitioned Gray code for the set of $q$-ary strings avoiding a pattern constituted by $k$ consecutive equal symbols. The definition of this Gray code is based on two different constructions, according to the parity of $q$. This result generalizes, and is based on, a Gray code for binary strings avoiding $k$ consecutive 0’s.

**Keyword**: Gray codes, pattern avoiding strings, generalized Fibonacci numbers

1 Introduction

The famous $k$-generalized Fibonacci sequence $\{f_n^{(k)}\}_{n \geq 0}$ is defined by

$$f_n^{(k)} = f_{n-1}^{(k)} + f_{n-2}^{(k)} + \cdots + f_{n-k}^{(k)} , \text{ for } n \geq k,$$

with initial conditions $f_i^{(k)} = 0$ for $0 \leq i \leq k-2$, and $f_{k-1}^{(k)} = 1$. This sequence is related to the enumeration of binary strings avoiding $k$ consecutive 1’s [4], called $k$-generalized Fibonacci strings.

If an alphabet of cardinality $q \geq 2$ is used, then the enumeration of the strings of length $n$ and avoiding a pattern constituted by $k$ consecutive occurrences of a same symbol is given by

$$f_n^{(k)} = (q-1) \left( f_{n-1,q}^{(k)} + f_{n-2,q}^{(k)} + \cdots + f_{n-k,q}^{(k)} \right), \text{ for } n \geq k,$$

with $f_i^{(k)} = q^i$, for $0 \leq i \leq k-1$; and in particular $f_{n+1}^{(k)} = f_{n,2}^{(k)}$. This sequence is a particular case of the weighted $k$-generalized Fibonacci sequence (studied and used in [5] [7]) where all the weights are equal to $q-1$. Similarly to the binary case, by a $q$-ary $k$-generalized Fibonacci string we mean a string over a $q \geq 2$ letter alphabet $A$ and avoiding $k$ consecutive occurrences of a given symbol of $A$. For example, for $k = 3$ and $A = \{0,1,2,3\}$, 111 or 222 is avoided, but not both.
Once a class of objects is defined, often it could be of interest to list or generate them according to a particular criterion. A special way to do this is to generate the objects so that any two consecutive ones differ as little as possible, i.e., in a Gray code manner [2]. Such a particular code has already been proposed for (binary) \( k \)-generalized Fibonacci strings [9].

In the present work we provide a Gray code for the \( q \)-ary \( k \)-generalized Fibonacci strings, \( q \geq 2 \), which extends the approach in [9]. Our method gives a trace partitioned code where strings with the same trace are contiguous (for more details see Section 2). Analogously to the kind of the forbidden pattern considered in [9] (which is a sequence of \( k \) consecutive 1’s), here we consider the avoidance of \( k \) consecutive occurrences of a give symbol, not necessarily 1, of a \( q \)-ary alphabet.

If \( L \) is a set of strings over an alphabet \( A \), then by \( L \) we denote the ordered list where the strings of \( L \) are listed following a certain criterion. If the Hamming distance [3] between two successive elements of \( L \) is bounded by a constant, than \( L \) is called Gray code list. The notations we are going to use are defined below:

- if \( \alpha \) and \( \beta \) are two same length strings, then \( d_H(\alpha, \beta) \) denotes their Hamming distance;
- \( \overline{L} \) denotes the list obtained by covering \( L \) in reverse order;
- first(\( L \)) and last(\( L \)) are the first and the last element of \( L \), respectively, and clearly first(\( L \)) = last(\( \overline{L} \)), and last(\( L \)) = first(\( \overline{L} \));
- if \( u \) is a string, then \( u \cdot L \) is the list obtained by prepending \( u \) to each string in \( L \);
- if \( L \) and \( L' \) are two lists, then \( L \circ L' \) is the concatenation of the two lists, obtained by appending the elements of \( L' \) after the elements of \( L \).

2 Gray codes

The well known Binary Reflected Gray Code (BRGC) [2] can be generalized to an alphabet of cardinality greater than 2 [1, 10]. If \( A = \{a_0, a_1, \ldots, a_{q-1}\} \), then the list \( G_q^n \) of the length \( n \) strings over \( A \) is given by:

\[
G_q^n = \begin{cases} 
\lambda & \text{if } n = 0, \\
 a_0 \cdot G_q^{n-1} \circ a_1 \cdot \overline{G_q^{n-1}} \circ \cdots \circ a_{q-1} \cdot \overline{G_q^{n-1}} & \text{if } n > 0,
\end{cases}
\]  

(1)

where \( \overline{G_q^{n-1}} \) is \( G_q^{n-1} \) if \( q \) is even or \( \overline{G_q^{n-1}} \) if \( q \) is odd, and \( \lambda \) is the empty string. It is proved that \( G_q^n \) is a Gray code list with Hamming distance 1 and the reader can easily prove the following proposition.

**Proposition 1.** If \( q \) is odd, then \( \text{last}(G_q^n) = a_{q-1}^n \) and \( \text{first}(G_q^n) = a_0^n \). In particular,

\[
\text{last}(G_q^n) = \text{last}(G_q^{n-1}) \cdot a_{q-1} = a_{q-1} \cdot \text{last}(G_q^{n-1}),
\]

and

\[
\text{first}(G_q^n) = \text{first}(G_q^{n-1}) \cdot a_0 = a_0 \cdot \text{first}(G_q^{n-1}).
\]
If \( q \) is even, then \( \text{last}(G_n^q) = a_{q-1}a_0^{n-1} \) and \( \text{first}(G_n^q) = a_0^n \). In particular, for \( n > 1 \),

\[
\text{last}(G_n^q) = a_{q-1}a_0^{n-1} = \text{last}(G_{n-1}^q) \cdot a_0,
\]

and

\[
\text{first}(G_n^q) = a_0^n = \text{first}(G_{n-1}^q) \cdot a_0 = a_0 \cdot \text{first}(G_{n-1}^q).
\]

We will give a Gray code for the set of \( q \)-ary \( k \)-generalized Fibonacci strings of length \( n \), where the Hamming distance between two consecutive strings is 1. The definition of our Gray code depends on the parity of \( q \), and we will refer simply to Gray code for a list where two successive elements have Hamming distance equal to 1, and without any loss of generality we will consider the alphabet \( A = \{0, 1, \ldots, q - 1\} \). For our topics we are going to consider the avoidance of the pattern \( 0^k \), for a given \( k \geq 2 \), but all of our constructions can be easily translated to the pattern \( i^k \), for any \( i \in A \).

Our approach starts from the definition of the Gray code for binary \( k \)-generalized Fibonacci strings given in [9], then using a bit replacing technique we extend the binary alphabet to \( A \), leading to a trace partitioned Gray code (where strings with the same trace are consecutive).

The trace of a \( q \)-ary string is a binary string obtained by replacing each symbol different from 0 by 1.

Before going along in our discussion, we define a tool for manipulating the symbols of a string. For \( q \geq 3 \), we denote by \( G_{t-1}^q \oplus 1 \) the list obtained from \( G_{t-1}^q \) by adding 1 to each symbol in each string in \( G_{t-1}^q \). Actually, \( G_{t-1}^q \oplus 1 \) is the Gray code defined in relation (1) for \( \{a_0, a_1, \ldots, a_{q-1}\} = \{1, 2, \ldots, q\} \). For example, \( G_2^3 \oplus 1 = (111, 112, 122, 121, 221, 222, 212, 211) \).

If \( \beta \) is a binary string of length \( n \) such that \( |\beta|_1 = t \) (the number of 1’s in \( \beta \)), we define the expansion of \( \beta \), denoted by \( \varepsilon(\beta) \), as the list of \( (q - 1)^t \) strings, where the \( i \)-th string is obtained by replacing the \( t \) 1’s of \( \beta \) by the \( t \) symbols (read from left to right) of the \( i \)-th string in \( G_{t-1}^q \oplus 1 \). For example, if \( q = 3 \) and \( \beta = 01011 \) (the trace), then with \( G_2^3 \oplus 1 \) given above, we have \( \varepsilon(\beta) = (01011, 01012, 01022, 01021, 02021, 02022, 02012, 02011) \). Notice that in particular \( \text{first}(\varepsilon(\beta)) = \beta \) and all the elements of \( \varepsilon(\beta) \) have the same trace.

We observe that \( \varepsilon(\beta) \) is the list obtained from \( G_{t-1}^q \) by adding 1 to each symbol of each string in \( G_{t-1}^q \), then inserting some 0’s, each time in the same positions. Since \( G_{t-1}^q \) is a Gray code and the insertions of the 0’s does not change the Hamming distance between two successive element of \( \varepsilon(\beta) \) (which is 1), we have the following:

**Proposition 2.** For any \( q \geq 3 \) and binary string \( \beta \), the list \( \varepsilon(\beta) \) is a Gray code.

The Gray code we are going to consider as the starting point of our argument is the one defined in [9], where the author deals with binary strings avoiding \( k \) consecutive 1’s. Since we are interested in the avoidance of \( k \) consecutive 0’s, we recall, for the sake of clearness, the definition in [9] adapted according to our needs which cause some slight differences with respect to the original definition in [9].

Let \( \mathcal{F}_n^{(k)} \) be the list defined as:

\[
\mathcal{F}_n^{(k)} = \begin{cases} 
\mathcal{C}_n & \text{if } 0 \leq n < k, \\
1 \cdot \mathcal{F}_{n-1}^{(k)} \circ 01 \cdot \mathcal{F}_{n-2}^{(k)} \circ 001 \cdot \mathcal{F}_{n-3}^{(k)} \circ \ldots \circ 0^{k-1}1 \cdot \mathcal{F}_{n-k}^{(k)} & \text{if } n \geq k,
\end{cases}
\]
where

\[ C_n = \begin{cases} 
\lambda & \text{if } n = 0, \\
1 \cdot C_{n-1} \circ 0 \cdot C_{n-1} & \text{if } n \geq 1,
\end{cases} \]

with \( \lambda \) the empty string.

It is proved \[9\] that \( F_{n,q}^{(k)} \) is a Gray code for the set of binary length \( n \) strings avoiding \( k \) consecutive 0’s, and thus the number of strings in \( F_{n,q}^{(k)} \) is \( f_{n+k}^{(k)} \), the \((n+k)\)-th value of the \( k \)-generalized Fibonacci sequence.

Now let \( F_{n,q}^{(k)} \) be the set of length \( n \) strings over \( A = \{0,1,\ldots,q - 1\} \), \( q > 2 \), avoiding \( k \) consecutive 0’s, so \( |F_{n,q}^{(k)}| = f_{n,q}^{(k)} \) (see Introduction). The aim is the construction of a Gray code for \( F_{n,q}^{(k)} \). Our definition of such a Gray code is based on the expansion \( \varepsilon(\alpha_i) \) (or its reverse \( \overline{\varepsilon(\alpha_i)} \)) of each element of \( F_{n,q}^{(k)} = (\alpha_1, \alpha_2, \ldots, \alpha_{f_{n+k}^{(k)}}) \), and then concatenating them opportunely, according to the parity of \( q \).

Let us illustrate the construction of the list for the set \( F_{n,q}^{(k)} \), of the form

\[ \varepsilon(\alpha_1) \circ \varepsilon(\alpha_2) \circ \varepsilon(\alpha_3) \circ \varepsilon(\alpha_4) \circ \cdots \quad (3) \]

where \( \alpha_i \) covers the list \( F_{n,q}^{(k)} \). As we will see below, this construction yields a Gray code when \( q \) is even, but not necessarily when \( q \) is odd.

When \( k = q = n = 3 \), the list \( F_{3,3}^{(k)} \) is \( F_{3,3}^{(3)} = (100, 101, 111, 110, 010, 011, 001) \); \( G_2^1 \oplus 1 = (1, 2) \), \( G_2^2 \oplus 1 = (11, 12, 22, 21) \), and \( G_2^2 \oplus 1 \) is given in the example preceding Proposition \[2\]. The expansions of the elements of \( F_{3,3}^{(3)} \) are:

\[
\begin{align*}
\varepsilon(\alpha_1) &= \varepsilon(100) = (100, 200) \\
\varepsilon(\alpha_2) &= \varepsilon(101) = (101, 102, 202, 201) \\
\varepsilon(\alpha_3) &= \varepsilon(111) = (111, 112, 122, 121, 222, 221, 212, 211) \\
\varepsilon(\alpha_4) &= \varepsilon(110) = (110, 120, 220, 210) \\
\varepsilon(\alpha_5) &= \varepsilon(010) = (010, 020) \\
\varepsilon(\alpha_6) &= \varepsilon(011) = (011, 012, 022, 021) \\
\varepsilon(\alpha_7) &= \varepsilon(001) = (001, 002),
\end{align*}
\]

and the list of the form \( (3) \) for \( F_{3,3}^{(3)} \) is

\[ \varepsilon(\alpha_1) \circ \varepsilon(\alpha_2) \circ \varepsilon(\alpha_3) \circ \varepsilon(\alpha_4) \circ \varepsilon(\alpha_5) \circ \varepsilon(\alpha_6) \circ \varepsilon(\alpha_7) \]

which is a Gray code, as it easily can be checked. However, this is not true in general. For example, if \( n = 4 \), \( k = q = 3 \), concatenating \( \varepsilon(\alpha_i) \) and \( \varepsilon(\alpha_{i+1}) \), alternatively as in \( (3) \) does not yield a Gray code. Indeed,

- \( \alpha_7 = 1100 \) and \( \alpha_8 = 0100 \), and

- \( \varepsilon(\alpha_7) = (1100, 1200, 2200, 2100) \) and \( \varepsilon(\alpha_8) = (0100, 0200) \).

And in the concatenation

\[ \varepsilon(\alpha_1) \circ \varepsilon(\alpha_2) \circ \varepsilon(\alpha_3) \circ \varepsilon(\alpha_4) \circ \varepsilon(\alpha_5) \circ \varepsilon(\alpha_6) \circ \varepsilon(\alpha_7) \circ \varepsilon(\alpha_8) \circ \varepsilon(\alpha_9) \circ \varepsilon(\alpha_{10}) \circ \varepsilon(\alpha_{11}) \circ \varepsilon(\alpha_{12}) \circ \varepsilon(\alpha_{13}), \]

in the transition from \( \varepsilon(\alpha_7) \) to \( \varepsilon(\alpha_8) \) we found 2100 followed by 0200 which differ in more than one position.
2.1 The case of \( q \) odd

A way to overcome the previous difficulties is to consider the partition of \( F_n^{(k)} \) as in its definition \(^2\). For \( j = 1, 2, \ldots, k \), let \( \alpha_i^{(j)} \) be the \( i \)-th element in the list \( 0^{j-1} \cdot F_{n-j}^{(k)} \), and

\[
\Gamma_j = \varepsilon(\alpha_1^{(j)}) \circ \varepsilon(\alpha_2^{(j)}) \circ \varepsilon(\alpha_3^{(j)}) \circ \ldots \circ \varepsilon(\alpha_\ell^{(j)}),
\]

where \( \varepsilon(\alpha_i^{(j)}) = \varepsilon(\alpha_i^{(j)\prime}) \) if \( f_{n+k-j}^{(k)} \) is odd and \( \varepsilon(\alpha_i^{(j)}) = \varepsilon(\alpha_i^{(j)\prime}) \) if \( f_{n+k-j}^{(k)} \) is even.

Let define \( F_{n,q}^{(k)} \) as

\[
F_{n,q}^{(k)} = \Gamma_1 \circ \Gamma_2 \circ \ldots \circ \Gamma_k,
\]

and clearly \( F_{n,q}^{(k)} \) is a list for the set \( F_{n,q}^{(k)} \) and the next proposition proves that it is a Gray code.

**Proposition 3.** If \( q \) is odd, then the list \( F_{n,q}^{(k)} = \Gamma_1 \circ \Gamma_2 \circ \ldots \circ \Gamma_k \) is a Gray code list with Hamming distance 1.

**Proof.** We have to prove the following:

1. \( \Gamma_j \) is a Gray code list, for each \( j = 1, 2, \ldots, k \), with Hamming distance 1;
2. \( d_H(\text{last}(\Gamma_j), \text{first}(\Gamma_{j+1})) = 1 \), for \( j = 1, 2, \ldots, k-1 \).

By Proposition \(^2\) it follows that \( \varepsilon(\alpha_i^{(j)}) \) is a Gray code, and for the point 1 we have to check that, if \( i \) is odd, \( d_H(\text{last}(\varepsilon(\alpha_i^{(j)})), \text{first}(\varepsilon(\alpha_{i+1}^{(j)}))) = 1 \) and that, if \( i \) is even, \( d_H(\text{last}(\varepsilon(\alpha_i^{(j)})), \text{first}(\varepsilon(\alpha_{i+1}^{(j)}))) = 1 \).

- When \( i \) is odd we observe that, for some \( v \) and \( w \),
  \[
  \alpha_i^{(j)} = 0^{j-1}v, \quad \text{and} \quad \alpha_{i+1}^{(j)} = 0^{j-1}w,
  \]
  where \( v \) and \( w \) differ in a single position since \( \alpha_i^{(j)} \) and \( \alpha_{i+1}^{(j)} \) are two consecutive binary strings in \( F_{n,q}^{(k)} \), which is a Gray code list.

Let \( t = |v|_1 \), and since \( q-1 \) is even, by Proposition \(^4\) it follows that \( \text{last}(G^{q-1}_{t+1}) = (q-1)1^t \) which occurs in the last element of the expansion of \( \alpha_i^{(j)} \). Therefore, \( \text{last}(\varepsilon(\alpha_i^{(j)})) = 0^{j-1}(q-1)v \).

Now, \( \text{first}(\varepsilon(\alpha_{i+1}^{(j)})) \) is equal to \( \text{last}(\varepsilon(\alpha_{i+1}^{(j)})) \), which as previously, is equal in turn to \( 0^{j-1}(q-1)w \).

Since \( v \) and \( w \) differ in a single position, so do \( \text{last}(\varepsilon(\alpha_i^{(j)})) \) and \( \text{first}(\varepsilon(\alpha_{i+1}^{(j)})) \).

- If \( i \) is even, and since \( \text{last}(\varepsilon(\alpha_i^{(j)})) = \text{first}(\varepsilon(\alpha_i^{(j)})) \), by the definition of expansion it follows that
  \[
  \text{first}(\varepsilon(\alpha_i^{(j)})) = \alpha_i^{(j)}, \quad \text{and} \quad \text{first}(\varepsilon(\alpha_{i+1}^{(j)})) = \alpha_{i+1}^{(j)}.
  \]
Since $\alpha_1^{(j)}$ and $\alpha_{i+1}^{(j)}$ are two consecutive strings, their Hamming distance is 1.

For the second point, we have

$$\text{first}(\Gamma_{j+1}) = \text{first}(\varepsilon(\alpha_{i+1}^{(j)})),$$

and

$$\text{last}(\Gamma_j) = \text{last}(\varepsilon(\alpha_i^{(j)})),$$

where, for some $w'$ and $w''$,

$$\alpha_1^{(j+1)} = 0^j w', \text{ and}$$

$$\alpha_i^{(j)} = 0^{j-1} w''.$$ 

Since $\alpha_i^{(j)}$ and $\alpha_{1}^{(j+1)}$ are two consecutive strings in $\mathcal{F}_n^{(k)}$, their Hamming distance is 1, and thus $w'' = 1 w'$. Two cases can occur:

- if $f_{n+k-j}^{(k)}$ is even, then $\text{last}(\Gamma_j) = \text{last}(\varepsilon(\alpha_i^{(j)})) = \text{last}(\varepsilon(\alpha_i^{(j)})) = \alpha_i^{(j)}$. Moreover, $\text{first}(\Gamma_{j+1}) = \text{first}(\varepsilon(\alpha_1^{(j+1)})) = \alpha_1^{(j+1)}$.

- if $f_{n+k-j}^{(k)}$ is odd, then $\text{last}(\Gamma_j) = \text{last}(\varepsilon(\alpha_i^{(j)})) = 0^j (q - 1) w''$ and $\text{first}(\Gamma_{j+1}) = \text{first}(\varepsilon(\alpha_1^{(j+1)})) = \alpha_1^{(j+1)} = 0^j 1 w' = 0^j w''$.

In any case, $d_H(\text{last}(\Gamma_j), \text{first}(\Gamma_{j+1})) = 1$.

Therefore the proof is concluded and $\mathcal{F}_n^{(k)}$ is a Gray code list with Hamming distance 1. \(\square\)

It is easy to see that, generally, when $q$ is even, the construction given in the previous proposition does not yield a Gray code.

For the sake of clearness, we illustrate the previous construction for the Gray code when $n = 4$, $k = 3$ and $A = \{0, 1, 2\}$. We have:

$$\mathcal{F}_4^{(3)} = \{(0101, 1011, 1010, 1110, 1111, 1101, 1100, 0100, 0101, 0111, 0110, 0010, 0011)\};$$

$$G_0^2 \oplus 1 = \lambda;$$

$$G_1^2 \oplus 1 = (1, 2);$$

$$G_2^2 \oplus 1 = (11, 12, 22, 21);$$

$$G_3^2 \oplus 1 = (111, 112, 122, 121, 221, 222, 212, 211);$$

$$G_4^2 \oplus 1 = (1111, 1112, 1122, 1121, 1221, 1222, 1212, 1211, 2211, 2212, 2222, 2221, 2121, 2122, 2112, 2111);$$

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\[ \Gamma_1 = (1001, 1002, 2002, 2001, 2111, \ldots, 1011, 1010, \ldots, 2110, \ldots, 1110), \]
\[ 1111, \ldots, 2111, 2101, \ldots, 1101, 1100, \ldots, 2100); \]
\[ \varepsilon(\alpha_1^{(1)}) \varepsilon(\alpha_2^{(1)}) \varepsilon(\alpha_3^{(1)}) \varepsilon(\alpha_4^{(1)}) \]
\[ \Gamma_2 = (0100, 0200, 0201, \ldots, 0101, 0111, \ldots, 0211, 0210, \ldots, 0110); \]
\[ \varepsilon(\alpha_2^{(2)}) \varepsilon(\alpha_3^{(2)}) \varepsilon(\alpha_4^{(2)}) \]
\[ \Gamma_3 = (0010, 0020, 0021, \ldots, 0011). \]
\[ \varepsilon(\alpha_3^{(3)}) \varepsilon(\alpha_4^{(3)}) \]

The reader can easily check that \( \mathcal{F}_{4,3}^{(3)} = \Gamma_1 \circ \Gamma_2 \circ \Gamma_3 \) is a Gray code with Hamming distance 1.

2.2 The case of \( q \) even

In this case the construction of a Gray code list for \( F_{n,q}^{(k)} \) is straightforward, and based on the discussion at the beginning of this section: just consider the expansions of the binary strings \( \alpha_i \) in \( F_{n,q}^{(k)} \), for \( i = 1, 2, \ldots, f_{n+k}^{(k)} \), and concatenate them, taking \( \varepsilon(\alpha_i) \) and \( \varepsilon(\alpha_{i+1}) \) alternatively, as in expression (3). The next proposition shows that the obtained list is a Gray code.

**Proposition 4.** If \( q \) is even, then the list

\[ \mathcal{F}_{n,q}^{(k)} = \varepsilon(\alpha_1) \circ \varepsilon(\alpha_2) \circ \ldots \circ \varepsilon'(\alpha_{f_{n+k}^{(k)}}), \]

where \( \varepsilon'(\alpha_{f_{n+k}^{(k)}}) = \varepsilon(\alpha_{f_{n+k}^{(k)}}) \) if \( f_{n+k}^{(k)} \) is odd and \( \varepsilon'(\alpha_{f_{n+k}^{(k)}}) = \varepsilon(\alpha_{f_{n+k}^{(k)}}) \) if \( f_{n+k}^{(k)} \) is even, is a Gray code list with Hamming distance 1.

**Proof.** By Proposition 2 each \( \varepsilon(\alpha_i) \) is a Gray code list, and, we have to check that, if \( i \) is odd, \( d_H(\text{last}(\varepsilon(\alpha_i)), \text{first}(\varepsilon(\alpha_{i+1}))) = 1 \) and that, if \( i \) is even, \( d_H(\text{last}(\varepsilon(\alpha_i)), \text{first}(\varepsilon(\alpha_{i+1}))) = 1 \).

- In the first case (\( i \) is odd), by Proposition 1 and the definition of expansion, it follows that in \( \text{last}(\varepsilon(\alpha_i)) \) and in \( \text{last}(\varepsilon(\alpha_{i+1})) \) the symbols different from 0 are equal to \( q-1 \). For example, if \( \alpha_i = 10110 \), then \( \text{last}(\varepsilon(\alpha_i)) = (q-1)(q-1)(q-1)0 \). Moreover, since \( \alpha_i \) and \( \alpha_{i+1} \) are two consecutive strings of \( \mathcal{F}_{n,q}^{(k)} \), \( d_H(\alpha_i, \alpha_{i+1}) = 1 \), and so \( d_H(\text{last}(\varepsilon(\alpha_i)), \text{last}(\varepsilon(\alpha_{i+1}))) = 1 \).

- If \( i \) is even, we observe that

\[ \text{last}(\varepsilon(\alpha_i)) = \text{first}(\varepsilon(\alpha_i)) = \alpha_i, \]
\[ \text{first}(\varepsilon(\alpha_{i+1})) = \alpha_{i+1}. \]

Since \( d_H(\alpha_i, \alpha_{i+1}) = 1 \), then \( d_H(\text{last}(\varepsilon(\alpha_i)), \text{first}(\varepsilon(\alpha_{i+1}))) = 1 \).

\( \square \)
3 Conclusions and further developments

In this paper we propose a trace partitioned Gray code for the $q$-ary $k$-generalized Fibonacci strings of length $n$, where the Hamming distance between two successive strings is 1. Our constructions are based on the expansion of an existing Gray code when $q = 2$. A consequence of the expansion technique is that our Gray code has the following property: if we replace each non-zero symbol in each string by 1, and ‘collapse’ the obtained list by keeping one copy of each binary strings, then the existing Gray code for $q = 2$ is obtained.

The investigation on the existence of Gray codes for strings on a $q$-ary alphabet avoiding a general consecutive pattern has already been studied: in [8] the author gives such a Gray code only when $q$ even, while the case of $q$ odd is left open. Our Gray code deals with the avoidance of a particular pattern but works for any $q$, and an interesting development could be a deeper investigation to check if this constructions can be applied to a general consecutive pattern in order to solve the open question in [8]. Also, it would be of interest to implement our Gray code definition into an efficient generating algorithm for the set of underlying strings.

References


